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A Simple Method of Radial Distortion Correction with Centre of Distortion Estimation

Aiqi Wang · Tianshuang Qiu · Longtan Shao

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Abstract This paper presents a new simple method to determine the distortion function of camera systems suffering from radial lens distortion. Neither information about the intrinsic camera parameters nor 3D-point correspondences are required. It is based on single image and uses the constraint, that straight lines in the 3D world project to circular arcs in the image plane, under the single parameter Division Model. Most of former approaches to correct the radial distortion are based on the collinearity of undistorted points. The proposed method in this paper, however, is based on the conclusion that distorted points are concyclic and uses directly the distorted points not undistorted points, therefore it should be more robust. It also computes the centre of radial distortion, which is important in obtaining optimal results. The results of experimental measurements on synthetic and real data are presented and discussed.

Keywords Radial distortion · Division model · Collinearity · RMS error

1 Introduction

Lens distortion is a significant problem in the analysis of digital images, especially 3D reconstruction and quantitative measurement. Usually, a ideal pinhole model is assumed in vary algorithms based on camera geometry. In reality, however, most lenses suffer from small or large amounts of distortion. When the lens has a nonnegligible distortion, using the ideal pinhole model may result in high measurement error.

The research on lens distortion can be traced back to 1919, when A. Conrady first introduced the decentering distortion model. Later, D.C. Brown presented the famous Brown-Conrady model in 1966 [1]. Since then this model has been widely used [2–5]. Lens distortion usually can be classified into three types: radial distortion, decentering distortion and thin prism distortion [6]. But in fact, for most lens, the radial component is predominant; to quote Zhang [3]: "it is likely that the distortion function is totally dominated by the radial components, and especially dominated by the first term. It has also been found that any more elaborated modeling not only would not help (negligible when compared with sensor quantization), but also would cause numerical instability".

Up to now, a number of methods were published which obtaining the parameters of the radial distortion function and correcting the images. These previous works can be divided roughly into two strategic approaches. The first is known as multiple views method which uses point correspondences of two or more images [7–13]. Stein [7] uses epipolar and trilinear constraints and searches for the amount of radial distortion that minimizes the errors in these constraints. Fizgibbon [8] proposes a linear method that simultaneously estimates the lens distortion and multiple view geometry such as fundamental matrices. Hartley and Kang [9] propose a method of correcting lens distortion in a parameter-free way. They also emphasize the necessity of determining the radial distortion center and give a novel algorithm to estimate

A. Wang (✉) · T. Qiu
School of Electronic and Information Engineering, Dalian University of Technology, Dalian 116024, China
e-mail: iqi79@yahoo.com.cn

T. Qiu
e-mail: qiutsh@dlut.edu.cn

L. Shao
Department of Engineering Mechanics, Dalian University of Technology, Dalian 116024, China

center of distortion. Micusik and Pajdla [10] study omnidirectional camera geometry, properties, and the possibility of their autocalibration from image correspondences. Their method generalizes Fizgibbon's technique to a wider class of cameras with a circular field of view. Since the multiple views method doesn't need to know especial condition in the scene, for example, straight line, there is a wide application range of this class of method. But the disadvantage of such method is that it needs multiple images which are sometimes not available.

This paper is in the second class of methods, which single view method based on the using of distorted straight lines [4, 14–20]. A fundamental property is often used: A camera follows the perspective camera model if and only if the projection of every 3D line in space onto the camera plane is a line. Consequently, one can estimate radial distortion parameter by measuring how much each line is distorted in the image. One common method is to compute the undistorted points using the corresponding distorted points, then to minimize the residual errors based on the collinearity of undistorted points. Almost all of them either suppose the distortion center is the image center or estimate it by iterative method. This class of methods requires only one image, therefore it is more flexible. The only precondition for line-based approaches is that the 3D scene contains straight 3D lines, which is valid for most man-made environments.

The big feature of our method that distinguishes it from most other correction approaches to radial distortion is that we study detailedly the geometric properties of a distorted "straight line" and the distortion center under radial distortion division model [8]. These properties not only enable a deep understanding of the lens distortion model, but also support the geometric constructions proposed for correction. We show that a 3D straight line in space can be distorted to a circle arc and the distortion center also lies on a circle which is concentric with the distorted "straight line". Our method to fully correct the lens radial distortion uses directly the distorted points not undistorted points, therefore it should be more robust. Fully correction is taken to mean estimating simultaneously the distortion parameter and distortion center. The importance of determining this point has been recognized in the photogrammetry community. Hartley and Kang [9] argue that the usual assumption that the distortion center is at the image center is not safe and "showed that for all cameras that we tried the distortion center was significantly displaced from the center of the image, by as much as 30 pixels in a 640 × 480 image".

The closest work to that we report here is that of Strand and Hayman [18]. They show that the locus of distorted points from a straight line is a circle arc and correct radial distortion by circle fitting, which is similar to our idea. However, they neglect the effect of the distortion center, and their assumption that the distortion center is at the image center is not safe [9].

Barreto and Daniilidis [11] also state that a straight line can be imaged into a circle when the radial distortion is modeled by the division model. But this principle only is used to classify the corresponding points as inliers or outliers, not to estimate the distortion center and distortion parameter.

This paper is organized as follows. Section 2 describes the radial distortion model and the calibration procedure. We firstly introduce the division model for distortion, secondly prove the graphics of distorted straight line is a circle, and finally describe the calibration procedure to estimate the center and the parameter of the radial distortion. Further study about the conditions only have one or two lines and non-square pixels model are discussed in Sect. 3. Section 4 provides the experimental results. Finally a conclusion is drawn in Sect. 5.

2 Model and Approach

2.1 Radial Distortion Models

Let (x_d, y_d) be the measurable coordinates of the distorted image point, (x_u, y_u) the coordinates of the undistorted image point. The Polynomial Model (PM) that most commonly used to describe radial distortion can be written as

$$r_u = r_d(1 + \lambda_1 r_d^2 + \lambda_2 r_d^4 + \cdots) \tag{1}$$

where r_d and r_u are the distances of the distorted point (x_d, y_d) and the undistorted point (x_u, y_u) to the distorted centre P respectively, and λ_i is the parameter of the radial distortion.

PM model works best for lens with small distortions. But when the distortion is large, it may be necessary to take into account many terms than practical.

Fitzgibbon [8] suggested the Division Model (DM) as

$$r_u = \frac{r_d}{1 + \lambda_1 r_d^2 + \lambda_2 r_d^4 + \cdots} \tag{2}$$

The most remarkable advantage of DM over PM is that it is able to express high distortion at much lower order. In particular, for many cameras one parameter suffices [8, 12].

Following, we use single parameter Division Model

$$r_u = \frac{r_d}{1 + \lambda r_d^2} \tag{3}$$

as our distortion model. In order to simplify the equation we suppose that the distorted centre P is the origin of the image coordinates system. Thus for the image coordinates it holds

$$x_u = \frac{x_d}{1 + \lambda r_d^2}, \qquad y_u = \frac{y_d}{1 + \lambda r_d^2} \tag{4}$$

where $r_d^2 = x_d^2 + y_d^2$.

2.2 The Figure of Distorted Straight Line

It is a most intuitive approach to investigate the distortion of an image that is to observe the image of a straight line whether it is still straight.

In the following we consider collinear points and their distorted images. Let $y = kx + b$ be the equation of a straight line, where k is the slope and b the y-intercept, namely, undistorted equation is $y_u = kx_u + b$. From (4), we have

$$\frac{y_d}{1+\lambda r_d^2} = k\frac{x_d}{1+\lambda r_d^2} + b \tag{5}$$

After reformulation we obtain

$$y_d = kx_d + b + b\lambda(x_d^2 + y_d^2) \tag{6}$$

That is

$$x_d^2 + y_d^2 + \frac{k}{b\lambda}x_d - \frac{1}{b\lambda}y_d + \frac{1}{\lambda} = 0 \tag{7}$$

Equation (7) is a circle equation, which shows the graphics of distorted "straight line" is a circle under the condition of model (3).

2.3 Estimate the Centre and Parameters of the Radial Distortion

Until now, we suppose that the distorted centre P is the origin of the image coordinates system. In this section, we let (x_0, y_0) be the coordinates of the distorted centre P. From (7), we have

$$(x_d - x_0)^2 + (y_d - y_0)^2 + \frac{k}{b\lambda}(x_d - x_0) - \frac{1}{b\lambda}(y_d - y_0) + \frac{1}{\lambda} = 0 \tag{8}$$

After reformulation we obtain

$$x_d^2 + y_d^2 + \left(\frac{k}{b\lambda} - 2x_0\right)x_d + \left(-\frac{1}{b\lambda} - 2y_0\right)y_d + x_0^2 + y_0^2 - \frac{k}{b\lambda}x_0 + \frac{1}{b\lambda}y_0 + \frac{1}{\lambda} = 0 \tag{9}$$

Let

$$A = \frac{k}{b\lambda} - 2x_0$$

$$B = -\frac{1}{b\lambda} - 2y_0$$

$$C = x_0^2 + y_0^2 - \frac{k}{b\lambda}x_0 + \frac{1}{b\lambda}y_0 + \frac{1}{\lambda}$$

then from (9), we have

$$x_d^2 + y_d^2 + Ax_d + By_d + C = 0 \tag{10}$$

and based on the relation of A, B, and C, we have

$$x_0^2 + y_0^2 + Ax_0 + By_0 + C - \frac{1}{\lambda} = 0 \tag{11}$$

Equation (10) indicates that a group of parameter (A, B, C) can be determined by fitting a circle to a "straight line" which extracted from the image, consequently we obtain a constraint equation (11) of distorted centre P. So theoretically the coordinates (x_0, y_0) of distorted centre P would be obtained so long as we can extract three "straight line" from the image and determine three groups of parameter $(A_i, B_i, C_i)_{i=1,2,3}$, that is

$$\begin{cases} (A_1 - A_2)x_0 + (B_1 - B_2)y_0 + (C_1 - C_2) = 0 \\ (A_1 - A_3)x_0 + (B_1 - B_3)y_0 + (C_1 - C_3) = 0 \\ (A_2 - A_3)x_0 + (B_2 - B_3)y_0 + (C_2 - C_3) = 0 \end{cases} \tag{12}$$

Then substitute them into (11), we can obtain the parameter of the radial distortion as follows

$$\frac{1}{\lambda} = x_0^2 + y_0^2 + Ax_0 + By_0 + C \tag{13}$$

To sum up, the whole algorithm (algorithm 1) to estimate the centre and the parameter of the radial distortion works as follows:

1. Extract n $(n \geq 3)$ "straight line" l_i $(i = 1, 2, \ldots, n)$ from the image;
2. Determine parameter (A_i, B_i, C_i) by fitting every "straight line" l_i with a circle according to (10);
3. Calculate the centre (x_0, y_0) of the radial distortion according to (12);
4. Calculate the parameter λ of radial distortion according to (13).

It is a very important step to fit circle above algorithm. Since the data extracted from image are only short arcs, it is hard to reconstruct a circle from the incomplete data. Here we introduce two methods of circle fitting. More details about this work may be found in [21, 22].

2.3.1 *Direct Least Squares Method of Circle Fitting (LS)*

For each point (x_i, y_i) on the "straight line", (10) gives

$$(x_i, y_i, 1)\begin{pmatrix} A \\ B \\ C \end{pmatrix} = -({x_i}^2 + {y_i}^2) \tag{14}$$

Stacking equations from N points together gives

$$\mathbf{M}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \mathbf{b} \tag{15}$$

where $\mathbf{M}$ and $\mathbf{b}$ are $N \times 3$ and $N \times 1$ matrices respectively. Directly using linear least squares fit method, we can get

$$(A, B, C)^\top = (\mathbf{M}^\top \mathbf{M})^{-1} \mathbf{M}^\top \mathbf{b} \tag{16}$$

This algorithm works very fast and non-iterative, but it is sensitive to noise. We call this method that using (16) to fit circle and estimate radial distortion parameter as LS.

2.3.2 Levenberg-Marquardt Method of Circle Fitting (LM)

In order to stably and accurately fit circle to arbitrary short arcs, N. Chernov and C. Lesort [21] present a new algorithm. Now we only outline its main ideas.

Let the equation of a circle be

$$a_1(x^2 + y^2) + a_2 x + a_3 y + a_4 = 0 \tag{17}$$

which subject to the constraint

$${a_2}^2 + {a_3}^2 - 4a_1 a_4 = 1 \tag{18}$$

The distance from a point (x_i, y_i) to the circle can be expressed as

$$d_i = \frac{P_i}{1 + \sqrt{1 + 4a_1 P_i}} \tag{19}$$

where

$$P_i = a_1(x_i^2 + y_i^2) + a_2 x_i + a_3 y_i + a_4 \tag{20}$$

From (18), we can define an angular coordinate θ by

$$a_2 = \sqrt{1 + 4a_1 a_4}\cos\theta, \qquad a_3 = \sqrt{1 + 4a_1 a_4}\sin\theta \tag{21}$$

so that θ will replace a_2 and a_3. Now one can apply the standard Levenberg-Marquardt scheme to minimize the sum of squared distance $\mathcal{F} = \sum {d_i}^2$ in the three dimensional parameter space (a_1, a_4, θ).

This algorithm is stable and robust. We call this method that using Levenberg-Marquardt scheme to fit circle and estimate radial distortion parameter as LM.

3 Further Discussion

It is emphasized that must have n ($n \geq 3$) "straight lines" are available from the image in Algorithm 1. In this section, we relax the constraint and discuss the conditions of only one or two "straight line" and non-square pixels.

3.1 Only One Straight Line (L1)

When only one "straight line" is available, we have to suppose the distortion center P is the image center C and then calculate the distortion parameter λ by (13).

Such method used to estimate radial distortion parameter with single line can be referred to as L1.

3.2 Only Two Straight Lines (L2)

When only two "straight line" l_1 and l_2 are available, (12) becomes

$$(A_1 - A_2)x_0 + (B_1 - B_2)y_0 + (C_1 - C_2) = 0 \tag{22}$$

Equation (22) indicates that the distortion center (x_0, y_0) subjects to linear constraint. Although the position of distortion center can not be determined by above equation, one can select a suitable region D around the image center $C = (C_x, C_y)$ as the guessed value of distortion center. For any $P_i \in D$, calculating the distortion parameter λ_i by (13). In order to obtain the optimal estimation of the distortion center and distortion parameter, we define the deviation measure d_i as the sum of the perpendicular distance from all the corrected points to their corresponding corrected straight line, which is estimated by least squares linear regression. The best quality of distortion center P and distortion parameter λ are reached when the value for the measure d_i becomes minimal.

We call this method that using two lines to estimate distortion center and radial distortion parameter as L2, which is described as follows:

1. Extract "straight line" l_1 and l_2 from the image;
2. Determine parameter $(A_1, B_1, C_1)((A_2, B_2, C_2))$ by fitting the "straight line" $l_1(l_2)$ with a circle according to (10);
3. Select a suitable interval $I (I = (C_x - 30, C_x + 30)$ is suggested), for any $x_0^i \in I$, calculating y_0^i according to (22);
4. Calculate the distortion parameter $\lambda_i = \frac{1}{2}(\lambda_i^1 + \lambda_i^2)$ according to (13), for any $P_i = (x_0^i, y_0^i)$;
5. Calculate the corresponding corrected points (x_u^1, y_u^1) $((x_u^2, y_u^2))$, for any λ_i, P_i and all distorted points $(x_d^1, y_d^1) \in l_1 ((x_d^2, y_d^2) \in l_2)$ according to (4);
6. Let $[d, k] = \min M_i = \min(d_i^1 + d_i^2)$, then obtain the optimal estimation P_k and λ_k.

3.3 Non-square Pixels

If the pixels are not square, the distortion is still radial. Let (x_0, y_0) be the coordinates of the distorted centre P, α be the pixel aspect radio, consequently the distorted radius is given by $r_d^2 = (x_d - x_0)^2 + \alpha^2 (y_d - y_0)^2$.

From (8) we have

$$(x_d - x_0)^2 + \alpha^2(y_d - y_0)^2 + \frac{k}{b\lambda}(x_d - x_0) - \frac{\alpha}{b\lambda}(y_d - y_0) + \frac{1}{\lambda} = 0 \tag{23}$$

After reformulation we obtain

$$x_d^2 + \alpha^2 y_d^2 + \left(\frac{k}{b\lambda} - 2x_0\right)x_d + \left(-\frac{\alpha}{b\lambda} - 2\alpha^2 y_0\right)y_d$$

$$+x_0^2+\alpha^2 y_0^2-\frac{k}{b\lambda}x_0+\frac{\alpha}{b\lambda}y_0+\frac{1}{\lambda}=0 \tag{24}$$

Equation (24) is an ellipse equation, which shows the graphics of distorted "straight line" is an ellipse under the condition of model (3).

Similarly let

$$A=\frac{k}{b\lambda}-2x_0$$
$$B=-\frac{\alpha}{b\lambda}-2\alpha^2 y_0$$
$$C=x_0^2+\alpha^2 y_0^2-\frac{k}{b\lambda}x_0+\frac{\alpha}{b\lambda}y_0+\frac{1}{\lambda}$$

then we have

$$x_d^2+\alpha^2 y_d^2+Ax_d+By_d+C=0 \tag{25}$$

and

$$x_0^2+\alpha^2 y_0^2+Ax_0+By_0+C-\frac{1}{\lambda}=0 \tag{26}$$

From (25) and (26), we can estimate the centre and the parameter of the radial distortion by modifying simply the algorithm as described. We need only to replace a best fitted circle with a best fitted ellipse [23].

4 Experiments and Results

In this section, we present some results of experimental measurements on synthetic and real image data. Since the synthetic image provides exact knowledge of line positions, distortion center and distortion parameter, the precise quantitative evaluation of performance is possible. The performance on real images is shown to demonstrate the practical implementation of the technique.

4.1 Tests on Synthetic Data

A 640 × 480 image consisting of 5 lines is used as a test image. The lines are generated with random orientations and positions. Using the single parameter DM model (3) with known distortion parameter $\lambda_{true}=-1.0\times 10^{-6}$ and distortion center $P_{true}=(320,240)$ is the image center, the line points were distorted. To simulate errors in feature extraction, the location of each point was noised by a zero-mean Gaussian noise with standard deviation, σ, expressed in pixels. In our tests, σ is varied from 0 to 2.5 pixels, which represents a typical range in video and film imagery [8]. We then used our approach to estimate the distortion parameters from the noisy data.

The first experiment on synthetic data compared the methods (LS and LM) presented in this paper with the method CK, proposed by C. Brauer-Burchardt and K. Voss in [14], at various noise levels. Because of using the same DM model, it is comparable. Results from 30 random trials are shown in Fig. 1.

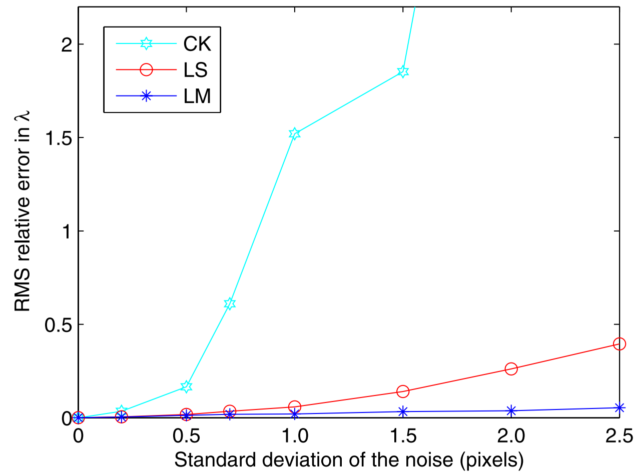

Fig. 1 Compare the robustness of three methods, LM, LS and CK, to estimate radial distortion parameter at various noise levels (pixels); RMS relative error in λ: $\frac{\lambda-\lambda_{true}}{\lambda_{true}}$

From Fig. 1, we can get that all of the three methods give good results about radial distortion parameter λ at very low noise levels. And LM method is very robust even at high noise levels. LS method is inferior to LM method. However, the performance of CK method is very poor when the noise levels become high. This maybe because that high noise levels badly destroy the collinearity of undistorted points.

The second experiment on synthetic data compared the methods LS, LM with the methods L1 and L2 using multiple lines (5 lines), one line and two lines respectively, where both methods L1 and L2 also are based on the Levenberg-Marquardt scheme. The noise level, σ, is varied from 0 to 2.5 pixels. Results from 30 random trials are shown in Figs. 2 and 3.

Figure 2 shows that these methods based on the Levenberg-Marquardt scheme are better than the LS method as well as Fig. 1, and that one could obtain better results with more lines. The estimation results of distortion center are depicted in Fig. 3. From this one can easily read out that all of three methods, LM, L2 and LS, give good estimation results about radial distortion center P at low noise levels. Even at the noise level, standard deviation $\sigma=1.0$, the relative error is still less than 6 pixels, which shows that the LM method are robust. The test also indicates that one can improve the estimation quality by using more lines.

The third experiment on synthetic data tested the performance of LM method and LS method at varying amount of distortion parameter, where λ varied at reasonable intervals from extreme barrel to extreme pin-cushion distortion and noise was constant at 1 pixel. As shown in Fig. 4 that the LM

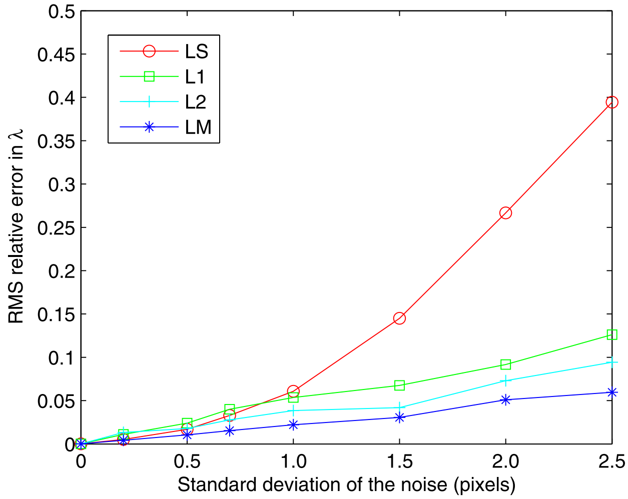

Fig. 2 Compare the robustness of four methods, LM, LS, L1 and L2, to estimate radial distortion parameter at various noise levels (pixels); RMS relative error in λ: $\frac{\lambda-\lambda_{true}}{\lambda_{true}}$

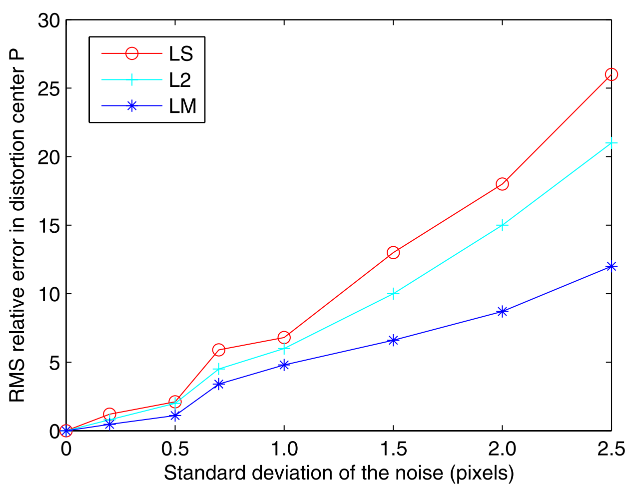

Fig. 3 Compare the robustness of three methods, LM, LS and L2, to estimate radial distortion center at various noise levels (pixels); RMS radial error in P: $\|P - P_{true}\|_2$

method works very well for almost all distortion parameter in the test interval. However, there is one point may need some more explanations. The LM method does not work very well when the distortion parameter λ closes to zero, which mainly because the amount of distortion is so small that it is affected greatly by the noise (1 pixel) added in the experiment. We have also applied LM method to some real fisheye lens images, the corrected results look very well.

4.2 Tests on Real Images

The next experiment is to correct radial distortion of real images. Since image distortion is sometimes less than a pixel, we used an edge detection method with a sub-pixel accuracy

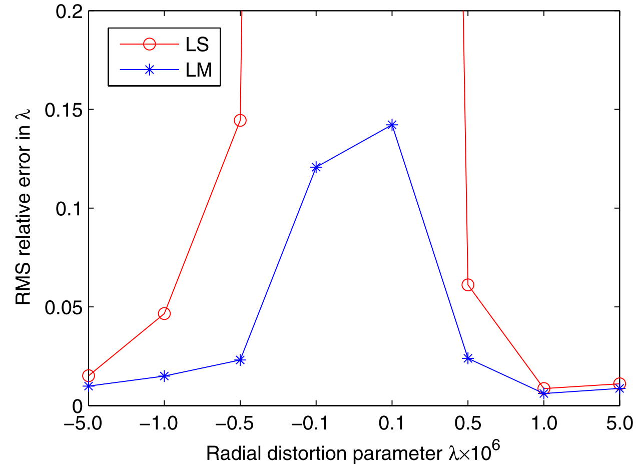

Fig. 4 Test the performance of LM method at various λ; The noise level, $\sigma = 1.0$ (pixels)

Table 1 Center (x_0, y_0) and parameter λ of the distorted image from Fig. 5

size	x_0/pixel	y_0/pixel	$\lambda \times 10^7$
640×480	302.5820	247.2381	−5.9566

[24], which is quite robust to noise. We used a threshold of about 50 on the length of the resulting edge chains because small segments may contain more noise than useful information about distortion. The detailed process to detect lines may be found in [4, 16].

The first image is a checkerboard with black-and-white squares (Fig. 5), from which, we can obviously observe the distortion.

Table 1 shows the estimated results of the centre and the parameter of the radial distortion applying our technique as described in Sect. 2. The center of radial distortion is (302.5820, 247.2381) (compared with the image center (320, 240)), and the parameter of the radial distortion is -5.9566×10^{-7}.

In order to better compare the results of correction, the corner points of distorted $(\cdot)$ and corrected $(+)$ are shown simultaneously in Fig. 6. Figure 7 shows the distortion corrected result of Fig. 5. As can be seen, the corrected result looks very reasonable.

The real image is also used to compare the method LM and the method CK (proposed in [14]) and the method DF (proposed in [4]). We have considered the collinearity of the corner points of every column, i.e., the RMS (root-mean-square) error in pixels between all the corrected points to their corresponding corrected straight line, as an accuracy measure. The results are shown in Table 2. There are 13 columns from right to left (except right most and left most) on the image. As shown in Table 2, that the accuracy of tested results both of the method LM and CK is similar in

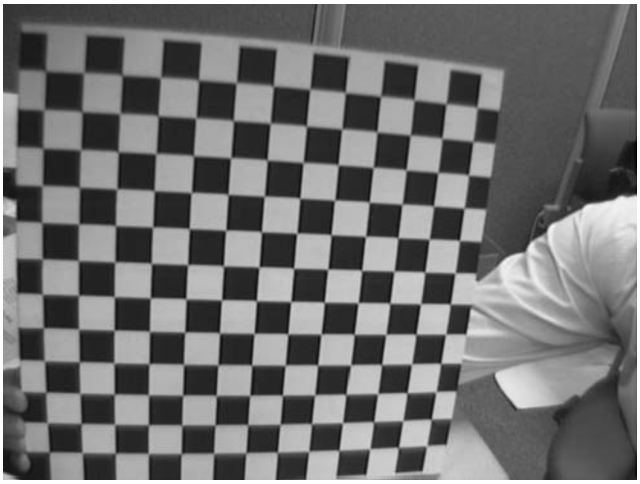

Fig. 5 An image suffering from lens distortion; The image size is 640 × 480

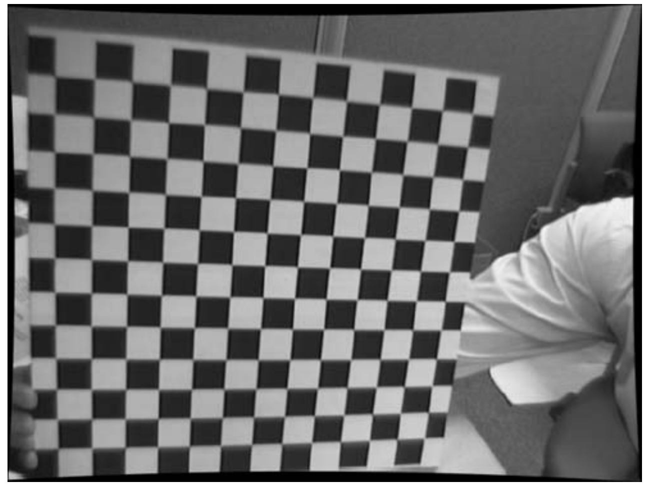

Fig. 7 Distortion correction result of Fig. 5

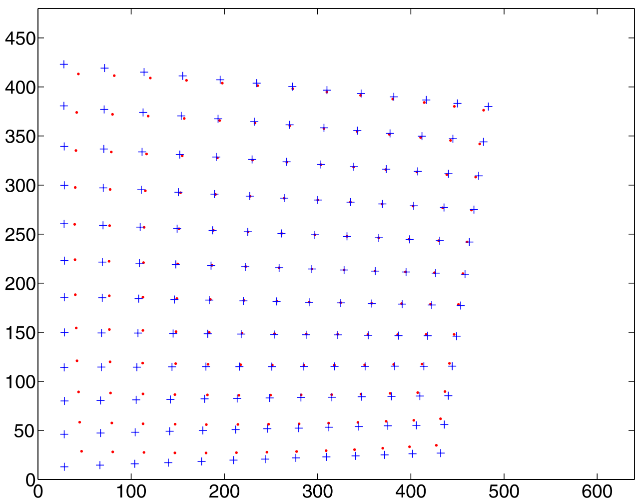

Fig. 6 The corner points of distorted (·) and corrected (+)

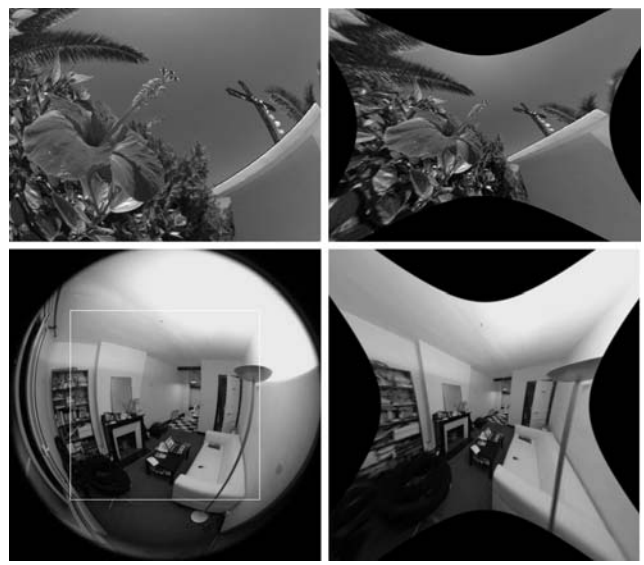

Fig. 8 Distortion removal results: input (*left column*) and corrected output (*right column*)

this experiment. But the LM method is more robust than CK method as shown in the Fig. 1.

Other distorted images and corrected results are shown in Fig. 8. On the top of Fig. 8, only one "straight line", which is illustrated in both of the input and the output images, is used. On the bottom of Fig. 8, the input is a fisheye image, and the part around the center of the input image is selected for showing the corrected result, because the size of the corrected image is too large to fit to show. As can be seen, the distortion removals are visible and the corrected results are very good.

5 Conclusions

In this paper, we present a new method to estimate the parameter of the radial distortion. Neither information about the intrinsic camera parameters nor 3D-point correspondences are required. It is based on single image and uses the distorted positions of collinear points. The algorithm is simple, robust and non-iterative. The big advantage of our method is that it works on a single image and simultaneously estimates the center and the parameter of the radial distortion. And we can also estimate the equations of straight line on the image (if needed). The disadvantage of our method is that it needs straight lines are available in the scene.

The presented method is suitable to correct distorted images, and the corrected result looks very reasonable.

Table 2 Collinearity of the corner points of every column

RMS	1	2	3	4	5	6	7	8	9	10	11	12	13
distorted	0.8806	0.7995	0.5667	0.3986	0.1599	0.1258	0.3464	0.5574	0.9011	1.1480	1.6616	5.9470	3.4686
DF	0.4298	0.3089	0.3024	0.2041	0.1949	0.1287	0.1119	0.2755	0.2971	0.4461	0.5200	0.6842	1.0100
CK	0.1018	0.1593	0.1207	0.1337	0.1327	0.1805	0.1051	0.1786	0.1185	0.1605	0.1621	0.1982	0.2795
LM	0.1066	0.0936	0.0996	0.0915	0.1415	0.1240	0.0804	0.1188	0.1081	0.1040	0.1171	0.1371	0.1886

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Aiqi Wang received the BS degree in Department of Mathematics, Liaoning Normal University, China, in 2002, the MS degree in Department of Mathematics, Dalian University of Technology, China, in 2005, and now is a doctoral student of School of Electronic and Information Engineering, Dalian University of Technology, Dalian. His research interests include pattern recognition, medical image processing and computer vision.

Tianshuang Qiu received the BS degree from Tianjin University, Tianjin, China, in 1983, the MS degree from Dalian University of Technology, Dalian, China, in 1993, and the Ph.D. from Southeastern University, Nanjing, China, in 1996, all in electrical engineering. He conducted his post-doctoral research in the Department of Electrical Engineering at Northern Illinois University, DeKalb, USA. He is currently a professor in School of Electronic and Information Engineering, Dalian University of Technology, Dalian. His research interests include signal processing, pattern recognition and computer vision.

Longtan Shao received the Ph.D. degree in Department of Engineering Mechanics, Dalian University of Technology, China, in 1996. He is currently a professor in Department of Engineering Mechanics.